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Linear Differential Equations with Delays: Admissibility and Conditional Exponential Stability

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1. INTRODUCTION

In Perron's classical paper on stability [5], a central concern is the relationship, for linear differential equations, between the condition that the nonhomogeneous equation has some bounded solution for every bounded "second member", on the one hand, and a certain form of conditional stability of the solutions of the homogeneous equation on the other. This idea was later extensively developed by Massera and Schäffer among others, their work having been collected in a monograph [3]. In a previous paper [1], the present authors examined linear difference equations and provided for them the analogues of the central results for differential equations in Ref. [3]. The important new difficulty encountered was, of course, the irreversibility of the process described by a difference equation, and new conceptual tools were developed to overcome it.

The present paper reports on an initial attempt at applying the same methods to the type of linear systems "next in order of complexity", viz., linear functional-differential equations, or linear differential equations with delay. Our method relies crucially on results in Ref. [1], but in order to make the concepts and results intelligible without excessive technicalities we restrict ourselves here to a rather special set of assumptions. In the unprinted report [2] we have given a much more general account, at the cost of increased technical density and greater reliance on Refs. [1] and [3] for terminology and notation; we have tried, however, to illustrate here the main ideas and the new nontrivial difficulties.

Specifically, we consider, on $[0, \infty)$, an equation of the form

$$\dot{u} + Lu + Mu = r \tag{1.1}$$

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and the corresponding homogeneous equation

$$\dot{u} + Lu + Mu = 0 \tag{1.2}$$

in a finite-dimensional Banach space E; r and L are continuous functions, vector- and operator-valued, respectively; the "solution" u is defined on $[-p, \infty)$ —p chosen as an integer for convenience—and M, the "memory functional", takes a continuous function u linearly into a continuous function Mu in such a way that the value of Mu at any given value of the argument t depends on the values of u at *preceding* values of t.

In fact, we shall restrict the "scope" of the memory still further: roughly speaking, M "remembers" only values of u at arguments that lag behind t by at least 1 (this "gap" has been normalized) and at most the fixed bound p. Among many other cases covered despite this restriction is the case of a finite number of fixed delays. The technical form adopted for this restriction (Section 4) avoids a statement on how the dependence of $Mu(t)$ on u varies locally with t; this allows the theory to cover such cases as that of a single continuously varying delay.

The assumptions of our main result (Theorem 7.3) are that L is bounded, that M transforms bounded functions "boundedly" into bounded functions, and that Eq. (1.1) has at least one bounded solution for each bounded r; this last condition may be expressed in the tradition of Refs. [1] and [3] by saying that the pair $(\mathbf{C}, \mathbf{C})$ is *admissible* for Eq. (1.1). The conclusion describes the behavior of the solutions of Eq. (1.2); in fact, it refers also to the solutions of the Eq. (1.2) restricted to an interval $[m, \infty)$ with integral m. It states, roughly, that all bounded solutions of these restricted equations tend uniformly exponentially to 0; that there exists a "complementary" *finite-dimensional* manifold of solutions of Eq. (1.2) that tend uniformly exponentially to infinity and "stay away" uniformly from the bounded solutions; and that bounded solutions of the restricted Eq. (1.2) and corresponding restrictions of solutions in the "complementary" manifold span *all* solutions of the restricted equation for each m. Such a behavior may well be termed, in the spirit of Ref. [1], an *exponential dichotomy* of the solutions of Eq. (1.2).

Rather than apply the methods of Refs. [1] and [3] afresh to Eqs. (1.1) and (1.2), we prefer to transform them into equivalent difference equations in a function space. Values of solutions of the difference equations correspond to "slices" of solutions of Eqs. (1.1) and (1.2), and the relevant properties of the latter equations are reflected in corresponding properties of the difference equations. It will be seen that the theory for difference equations in Ref. [1], with its built-in irreversibility, is sufficient to account for the behavior of slices of solutions of Eqs. (1.1) and (1.2). It is true that only slices with integral endpoints are primarily dealt with; we feel this blemish is minor, and it can in fact be removed in most respects with additional effort. The technical core of this "translation" into difference equations is Section 6. Section 7 then gives the main results by "retranslating" the corresponding theorems of Ref. [1].

In this paper we deal only with the "continuous case"; Ref. [2] gives a parallel account of the "Carathéodory case", where Eqs. (1.1) and (1.2) only hold locally in $\mathbf{L}^1$, and where boundedness is replaced by membership in translation-invariant spaces of measurable functions. Future efforts will be directed to removing the assumption of the short-range "gap" in the memory, and to formulating an adequate general concept of "dichotomy" and "exponential dichotomy".

Substantial results on admissibility and dichotomies for certain linear functional-differential equations have been obtained independently, using a different method, by G. Pecelli in a recent thesis (Johns Hopkins University, 1969). The results of this thesis, which was made available to the authors after the original version of the present work was completed, are to be published in a forthcoming paper [4].

2. Spaces

Throughout this paper, E will denote a given real or complex Banach space; in Section 7 we shall assume that its dimension is finite. The norm in E, as in all normed spaces for which no other symbol is prescribed, is denoted by $\| \ \|$. If X, Y are Banach spaces, $[X \to Y]$ denotes the Banach space of operators (bounded linear mappings) from X to Y, and we set $\mathfrak{X} = [X \to X]$.

We shall be dealing with sequences and with functions defined on intervals of the real line. We denote by ω the set $\{0, 1,...\}$ of all natural numbers, and set $\omega_{[m]} = \{n \in \omega : n \geqslant m\}$, $m = 0, 1,...$. The notation for intervals of the real line is the usual one.

If m, m' are real numbers [natural numbers] with $m' \geqslant m$, and f is a function defined on $[m, \infty)$ [on $\omega_{[m]}$], then $f_{[m']}$ shall denote the restriction of f to $[m', \infty)$ [to $\omega_{[m']}$].

Assume that X is a Banach space. For each natural number m we denote by $\mathbf{s}_{[m]}(X)$ the linear space of all functions $f : \omega_{[m]} \to X$ and by $\mathbf{l}^\infty_{[m]}(X)$ the Banach space of all bounded ones, with the norm $| f | = \sup\{\|f(n)\| : n \in \omega_{[m]}\}$. For each real m we denote by $\mathbf{K}_{[m]}(X)$ the linear space of all continuous functions $f : [m, \infty) \to X$ and by $\mathbf{C}_{[m]}(X)$ the Banach space of all bounded ones among them, with the norm $| f | = \sup\{\|f(t)\| : t \in [m, \infty)\}$. In all these notations the subscript is omitted when $m = 0$.

Finally, for each real $m > 0$ we denote by $\mathbf{E}_m$ the Banach space of all continuous functions $f : [-m, 0] \to E$, with the norm $\| f \| = \max\{\|f(t)\| : t \in [-m, 0]\}$. (In Ref. [2] this space was denoted by ${}_m\mathbf{C}(E)$.)

The following example illustrates some obvious notational conventions. Suppose that $g \in \mathbf{l}^\infty(\mathbf{E}_m)$; then $\| g \|$ is the element of $\mathbf{l}^\infty(R)$ given by $\| g \|(n) = \| g(n) \|$, $n = 0, 1,...$; and $| g | = | \| g \| |$ is the norm of g as an element of $\mathbf{l}^\infty(\mathbf{E}_m)$.

3. Slicing Operations

From now on and throughout the paper, p will always denote a fixed positive integer and m, m' will be used exclusively to denote nonnegative integers.

Let m be given. For each integer $n > m$ we define the linear mapping $\varpi(n) : \mathbf{K}_{[m]}(E) \to \mathbf{E}_1$ by

$$(\varpi(n)f)(t) = f(t + n) \qquad t \in [-1, 0], \quad f \in \mathbf{K}_{[m]}(E). \tag{3.1}$$

Thus $\varpi(n)$ maps f into the "slice" of f between $n - 1$ and n, transplanted to $[-1, 0]$ for convenience. Then ϖf denotes the sequence $(\varpi(n)f)$, i.e., the function on $\omega_{[m+1]}$ whose values are the slices of f: $(\varpi f)(n) = \varpi(n)f$, $n = m + 1, m + 2,...$. Thus ϖ is a linear and obviously injective mapping of $\mathbf{K}_{[m]}(E)$ into $\mathbf{s}_{[m+1]}(\mathbf{E}_1)$. What is its range? We record the obvious answer.

Lemma 3.1. *Assume m and $g \in \mathbf{s}_{[m+1]}(\mathbf{E}_1)$ given. Then $g = \varpi f$ for some $f \in \mathbf{K}_{[m]}(E)$ if and only if $(g(n))(0) = (g(n + 1))(-1)$, $n = m + 1, m + 2,...$; if so, then f is bounded if and only if g is bounded, and $| f | = | g |$.*

We require a similar operation yielding slices of length p. For given m and each integer $n \geqslant m$ we define the linear mapping $\Pi(n) : \mathbf{K}_{[m-p]}(E) \to \mathbf{E}_p$ by

$$(\Pi(n)f)(t) = f(t + n) \qquad t \in [-p, 0], \quad f \in \mathbf{K}_{[m-p]}(E). \tag{3.2}$$

In analogy to the preceding, we define $\Pi : \mathbf{K}_{[m-p]}(E) \to \mathbf{s}_{[m]}(\mathbf{E}_p)$ by $(\Pi f)(n) = \Pi(n)f$, $n = m, m + 1,...$. Thus, if $p = 1$ and $m > 0$, Π coincides with the operator ϖ on $\mathbf{K}_{[m-1]}(E)$.

4. The Memory Functional

We now make precise the assumptions on the "memory functional" M that appears in Eq. (1.1). It is, at all events, a linear mapping from $\mathbf{K}_{[-p]}(E)$ into $\mathbf{K}(E)$, and the crux of the conditions on M mentioned in the introduction is that, for any $u \in \mathbf{K}_{[-p]}(E)$, the slice of Mu between $n - 1$ and n depends only on the slice of u between $n - 1 - p$ and $n - 1$. An additional condition is that M maps bounded functions "boundedly" into bounded functions, i.e., that it has a restriction to a bounded linear mapping of $\mathbf{C}_{[-p]}(E)$ into $\mathbf{C}(E)$.

LEMMA 4.1. *Assume that* $M : \mathbf{K}_{[-p]}(E) \to \mathbf{K}(E)$ *is a linear mapping satisfying*

(M_1): *if* $n \in \omega_{[1]}$ *and* $u, u' \in \mathbf{K}_{[-p]}(E)$ *satisfy* $\Pi(n-1)u = \Pi(n-1)u'$, *then* $\varpi(n)(Mu) = \varpi(n)(Mu')$.
Then there exists a unique sequence $(\hat{M}(n))$ *of linear mappings* $\hat{M}(n) : \mathbf{E}_p \to \mathbf{E}_1$, $n = 0, 1,\ldots$, *such that*

$$(\varpi Mu)(n) = \varpi(n)(Mu) = \hat{M}(n-1)\Pi(n-1)u, \qquad n = 1, 2,\ldots, \quad u \in \mathbf{K}_{[-p]}(E). \tag{4.1}$$

If $v_1, v_2 \in \mathbf{E}_p$ *and* $v_1(t+1) = v_2(t)$, $-p \leqslant t \leqslant -1$, *then*

$$(\hat{M}(n-1)v_1)(0) = (\hat{M}(n)v_2)(-1), \qquad n = 1, 2,\ldots. \tag{4.2}$$

Proof. For every $n \in \omega_{[1]}$ and every $v \in \mathbf{E}_p$ there exists $u \in \mathbf{K}_{[-p]}(E)$ such that $\Pi(n-1)u = v$, and by (M_1) $\varpi(n)(Mu)$ is the same for all such u; thus $v \mapsto \varpi(n)(Mu)$ is a well-defined mapping $\hat{M}(n-1) : \mathbf{E}_p \to \mathbf{E}_1$, and it satisfies Eq. (4.1). It is obviously linear and uniquely determined by Eq. (4.1).

If $n \in \omega_{[1]}$ and v_1, v_2 are as in the second paragraph of the statement, there exists $u \in \mathbf{K}_{[-p]}(E)$ such that $\Pi(n-1)u = v_1$, $\Pi(n)u = v_2$. Since $Mu \in \mathbf{K}(E)$, Lemma 3.1 and Eq. (4.1) yield $(\hat{M}(n-1)v_1)(0) = (\hat{M}(n-1)\Pi(n-1)u)(0) = (\varpi Mu(n))(0) = (\varpi Mu(n+1))(-1) = (\hat{M}(n)\Pi(n)u)(-1) = (\hat{M}(n)v_2)(-1)$. This proves Eq. (4.2).

Remark. Conversely, if $(\hat{M}(n))$ is a sequence of linear mappings of $\mathbf{E}_p$ into $\mathbf{E}_1$ that satisfies Eq. (4.2), there exists a unique linear mapping $M : \mathbf{K}_{[-p]}(E) \to \mathbf{K}(E)$ satisfying Eq. (4.1), and this M satisfies (M_1). The simple proof, omitted here, is based on Lemma 3.1.

Assumption (M_1) permits, for every m, the "cutting down" of M to a linear mapping $M_{[m]} : \mathbf{K}_{[m-p]}(E) \to \mathbf{K}_{[m]}(E)$ by means of

$$\varpi(n)(M_{[m]}u) = \hat{M}(n-1)\Pi(n-1)u, \qquad n = m+1, m+2,\ldots, \quad u \in \mathbf{K}_{[m-p]}(E). \tag{4.3}$$

If $m' \geqslant m \geqslant 0$, these cut-down memory functionals then satisfy

$$M_{[m']}u_{[m'-p]} = (M_{[m]}u)_{[m']}, \qquad u \in \mathbf{K}_{[m-p]}(E). \tag{4.4}$$

LEMMA 4.2. *Assume that* M, $(\hat{M}(n))$ *are as in Lemma* 4.1. *Then the condition*

(M_2): *the restriction of* M *to* $\mathbf{C}_{[-p]}(E)$ *is a bounded linear mapping*

$$M_{\mathbf{C}} : \mathbf{C}_{[-p]}(E) \to \mathbf{C}(E)$$

holds if and only if each $\hat{M}(n)$ ***is bounded and the sequence*** $(\hat{M}(n))$ ***is bounded in*** $[\mathbf{E}_p \to \mathbf{E}_1]$, i.e., ***constitutes a function*** $\hat{M} \in \mathbf{l}^\infty([\mathbf{E}_p \to \mathbf{E}_1])$. ***In this case***

$$|\hat{M}| = \| M_{\mathbf{C}} \|. \tag{4.5}$$

Proof. For every $n \in \omega_{[1]}$ and every $v \in \mathbf{E}_p$ there exists $u \in \mathbf{C}_{[-p]}(E)$ with $\Pi(n-1)u = v$ and $|u| = \| v \|$. If (M_2) holds, we have by Eq. (4.1), $\| \hat{M}(n-1)v \| = \| \hat{M}(n-1)\Pi(n-1)u \| = \| \varpi(n)(Mu) \| \leqslant |Mu| \leqslant \| M_{\mathbf{C}} \| \, |u| = \| M_{\mathbf{C}} \| \, \| v \|$. Thus $\hat{M}(n-1)$ is uniformly bounded, $\hat{M} \in \mathbf{l}^\infty([\mathbf{E}_p \to \mathbf{E}_1])$, and $|\hat{M}| \leqslant \| M_{\mathbf{C}} \|$. Assume, conversely, that $\hat{M} = (\hat{M}(n))$ is a bounded sequence in $[\mathbf{E}_p \to \mathbf{E}_1]$. For each $u \in \mathbf{C}_{[-p]}(E)$, Eq. (4.1) yields $\| \varpi Mu(n) \| \leqslant \| \hat{M}(n-1) \| \, \| \Pi(n-1)u \| \leqslant |\hat{M}| \, |u|$, $n = 1, 2, \ldots$; by Lemma 3.1, $Mu \in \mathbf{C}(E)$ and $|Mu| \leqslant |\hat{M}| \, |u|$. Therefore, (M_2) holds and $\| M_{\mathbf{C}} \| \leqslant |\hat{M}|$.

If M satisfies (M_1) and (M_2), formula (4.1) may be rewritten, more compactly, as

$$(\varpi Mu)(n) = (\hat{M}\Pi u)(n-1), \qquad n = 1, 2, \ldots . \tag{4.6}$$

5. Solutions

We assume given the space E and the positive integer p. We further assume given the operator-valued function $L \in \mathbf{C}(\tilde{E})$ and the memory functional M satisfying conditions (M_1), (M_2).

For every $r \in \mathbf{K}(E)$, a *solution of* Eq. (1.1) is a function $u \in \mathbf{K}_{[-p]}(E)$ whose restriction $u_{[0]}$ is continuously differentiable (the derivative is $\dot{u}_{[0]} \in \mathbf{K}(E)$) and that satisfies $\dot{u}_{[0]} + Lu_{[0]} + Mu = r$ on $[0, \infty)$ (strictly speaking, Eq. (1.1) should be written this way). More generally, for every m, a *solution of* Eq. $(1.1)_{[m]}$ is a function $u \in \mathbf{K}_{[m-p]}(E)$ whose restriction $u_{[m]}$ is continuously differentiable and that satisfies $\dot{u}_{[m]} + L_{[m]}u_{[m]} + M_{[m]}u = r_{[m]}$ in $[m, \infty)$. In particular, if $m' \geqslant m \geqslant 0$ and u is a solution of Eq. $(1.1)_{[m]}$, then $u_{[m'-p]}$ is a solution of Eq. $(1.1)_{[m']}$, on account of Eq. (4.4). These definitions and statements of course also apply to the homogeneous Eq. (1.2).

We define $V \in \mathbf{K}(\tilde{E})$ as the solution of the operator equation $\dot{V} + LV = 0$ on $[0, \infty)$ that satisfies $V(0) = I$ (I is the identity on E). It is well-known [3, Section 31] that V is invertible-valued; we write $V^{-1} \in \mathbf{K}(\tilde{E})$ for the function satisfying $V^{-1}(t) = (V(t))^{-1}$, $t \in [0, \infty)$. We also have

$$\| V(t)\, V^{-1}(s) \| \leqslant \exp\left(\left| \int_s^t \| L(\sigma) \| \, d\sigma \right| \right) \leqslant \exp(|t - s| \, |L|), \qquad s, t \in [0, \infty). \tag{5.1}$$

With this notation we find, with the use of Eq. (4.6), that every solution of Eq. $(1.1)_{[m]}$ satisfies

$$
\begin{aligned}
(\Pi u(n))(t) &= u(t+n) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.2)\\
&= \begin{cases} u(t+1+n-1) = (\Pi u(n-1))(t+1), & -p \leqslant t \leqslant -1 \\ V(t+n)\, V^{-1}(n-1)\, u(n-1) \\ \quad - \displaystyle\int_{-1}^{t} V(t+n)\, V^{-1}(s+n)(M_{[m]}u(s+n) - r(s+n))\, ds \\ \quad = V(t+n)\, V^{-1}(n-1)(\Pi u(n-1))(0) \\ \quad - \displaystyle\int_{-1}^{t} V(t+n)\, V^{-1}(s+n)(\hat{M}\Pi u(n-1) - \varpi r(n))(s)\, ds, \\ & -1 \leqslant t \leqslant 0, \end{cases}\\
&\qquad n = m+1, m+2, \ldots;
\end{aligned}
$$

and, conversely, every function $u \in \mathbf{K}_{[m-p]}(E)$ such that Πu satisfies Eq. (5.2)—more precisely, the equality between leftmost and rightmost sides—is a solution of Eq. $(1.1)_{[m]}$. For $-1 \leqslant t \leqslant 0$ the solution is found by the usual "variation of constants" [3, Section 31].

6. The Associated Difference Equation

The relation (5.2) is a difference equation for Πu; we proceed to make explicit the form of this equation. For this purpose, we define $A \in \mathbf{s}_{[1]}(\tilde{\mathbf{E}}_p)$ and $B \in \mathbf{s}_{[1]}([\mathbf{E}_1 \to \mathbf{E}_p])$ as follows:

$$
(A(n)\, v)(t) = \begin{cases} -v(t+1), & -p \leqslant t \leqslant -1 \\ -V(t+n)\, V^{-1}(n-1)\, v(0) \\ \quad + \displaystyle\int_{-1}^{t} V(t+n)\, V^{-1}(s+n)(\hat{M}(n-1)\, v)(s)\, ds, & -1 \leqslant t \leqslant 0 \end{cases}
$$

$$
n = 1, 2, \ldots; \qquad v \in \mathbf{E}_p; \tag{6.1}
$$

$$
(B(n)\, g)(t) = \begin{cases} 0, & -p \leqslant t \leqslant -1 \\ \displaystyle\int_{-1}^{t} V(t+n)\, V^{-1}(s+n)\, g(s)\, ds, & -1 \leqslant t \leqslant 0 \end{cases} \tag{6.2}
$$

$$
n = 1, 2, \ldots; \qquad g \in \mathbf{E}_1 .
$$

We note that the functions $A(n)v$ and $B(n)g$ thus defined are indeed continuous, even at $t = -1$. Using Eq. (5.1) we find

$$\begin{aligned} \| A(n)\, v \| &\leqslant \max\{\| v \|, (\| v \| + \| \hat{M}(n-1)\| \| v \|) \exp | L |\} \\ &\leqslant \| v \| (1 + | \hat{M} |) \exp | L | \\ \| B(n)\, g \| &\leqslant \| g \| \exp | L |, \end{aligned}$$

so that $A(n)$, $B(n)$ are indeed operators, as claimed, and in fact

$$A \in \mathbf{l}^{\infty}_{[1]}(\tilde{\mathbf{E}}_p), \qquad | A | \leqslant (1 + | \hat{M} |) \exp | L | \tag{6.3}$$

$$B \in \mathbf{l}^{\infty}_{[1]}([\tilde{\mathbf{E}}_1 \to \mathbf{E}_p]), \qquad | B | \leqslant \exp | L | \tag{6.4}$$

We consider the difference equations in $\mathbf{E}_p$

$$x(n) + A(n)x(n-1) = f(n), \qquad n = 1, 2,... \tag{6.5}$$

$$x(n) + A(n)x(n-1) = 0, \qquad n = 1, 2,... \tag{6.6}$$

and their restrictions $(6.5)_{[m]}$, $(6.6)_{[m]}$ to $n = m + 1$, $m + 2,\ldots$. Here $f \in \mathbf{s}_{[1]}(\mathbf{E}_p)$.

LEMMA 6.1. *Let m and $r \in \mathbf{K}(E)$ be given. A functions $x \in \mathbf{s}_{[m]}(\mathbf{E}_p)$ is a solution of Eq. $(6.5)_{[m]}$ with $f = B\varpi r$ if and only if $x = \Pi u$ for some solution u of Eq. $(1.1)_{[m]}$. In particular, x is a solution of Eq. $(6.6)_{[m]}$ if and only if $x = \Pi u$ for some solution u of Eq. $(1.2)_{[m]}$.*

Proof. If u is a solution of Eq. $(1.1)_{[m]}$, then Πu satisfies Eq. (5.2); together with Eqs. (6.1) and (6.2) this implies that $\Pi u(n) + A(n)\Pi u(n-1) = B(n)\varpi r(n)$, $n = m + 1$, $m + 2,\ldots$, i.e., that Πu is a solution of Eq. $(6.5)_{[m]}$ with $f = B\varpi r$. Conversely, if x is a solution of Eq. $(6.5)_{[m]}$ with $f = B\varpi r$, Eq. (6.2) implies $(f(n))(t) = 0$, $-p \leqslant t \leqslant -1$, $n = m + 1$, $m + 2,\ldots$, and this together with Eq. (6.1) implies that $(x(n))(t) = (x(n-1))(t+1)$ for all such t, n; there exists, therefore, a continuous u, i.e., $u \in \mathbf{K}_{[m-p]}(E)$, such that $x = \Pi u$. Using again the fact that x is a solution of Eq. $(6.5)_{[m]}$ with $f = B\varpi r$, we conclude that Πu satisfies Eq. (5.2), so that u is a solution of Eq. $(1.1)_{[m]}$.

It is clear that not every $f \in \mathbf{s}_{[1]}(\mathbf{E}_p)$ is of the form $f = B\varpi r$. It is still possible, however, to relate Eq. (6.5) with arbitrary f to Eq. (1.1). We shall do this here for bounded f only; for the general case, see Ref. [2, Theorem 6.2].

THEOREM 6.2. *For each $f \in \mathbf{l}^{\infty}_{[1]}(\mathbf{E}_p)$, there exists $r \in \mathbf{C}(E)$ such that*

$$| r | \leqslant k_1 | f |, \tag{6.7}$$

and such that the solution w of

$$w(n) + A(n)w(n-1) = f(n) - B\varpi r(n), \qquad n = 1, 2,... \tag{6.8}$$

with $w(0) = 0$ is bounded and satisfies

$$| w | \leqslant k_2 | f |, \tag{6.9}$$

where k_1, $k_2 > 0$ depend on p, $| L |$, $| \hat{M} |$ only.

Proof. We define $g \in \mathbf{s}_{[1]}(\mathbf{E}_1)$ by

$$(g(n))(t) = -6t(1+t)(f(n))(0) + t(3t+2)(\hat{M}f(n))(-1),$$
$$-1 \leqslant t \leqslant 0, \qquad n = 1, 2,... .$$

Obviously,

$$g(n)(-1) = (\hat{M}f(n))(-1) \qquad g(n)(0) = 0, \qquad n = 1, 2,..., \tag{6.10}$$

$$\int_{-1}^{0} (g(n))(s) = (f(n))(0), \qquad n = 1, 2,... , \tag{6.11}$$

and $\| g(n) \| \leqslant \frac{3}{2} \| f(n) \| + \| \hat{M}f(n) \|$, so that

$$g \in \mathbf{l}^{\infty}_{[1]}(\mathbf{E}_1), \qquad | g | \leqslant (\tfrac{3}{2} + | \hat{M} |) | f |. \tag{6.12}$$

We extend f and g to all integral arguments (in fact $n = -p,..., 0, 1,...$ would be enough) by setting, in $\mathbf{E}_p$ and $\mathbf{E}_1$, respectively,

$$\begin{aligned} f_0(n) &= 0, \qquad & g_0(n) &= 0 \qquad & n &= 0, -1, -2,... \\ f_0(n) &= f(n), \qquad & g_0(n) &= g(n) \qquad & n &= 1, 2,... . \end{aligned} \tag{6.13}$$

We now define $w \in \mathbf{s}(\mathbf{E}_p)$ by

$$\begin{aligned} (w(n))(t) = {} & V(t+n)\, V^{-1}([t]+n) \int_{t-[t]-1}^{0} (g_0(n+[t])(s)\, ds \\ & + \sum_{i=0}^{-[t]-1} (f_0(n-i))(t+i), \qquad -p \leqslant t \leqslant 0, \quad n = 0, 1,..., \end{aligned} \tag{6.14}$$

where $[t]$ denotes the greatest integer $\leqslant t$.

Equation (6.14) indeed yields a continuous function $w(n)$, for by Eqs. (6.11) and (6.13) we have, at $k = 0,\ldots, p - 1$,

$$
\begin{aligned}
(w(n))(-k) &= V(n-k)\, V^{-1}(n-k) \int_{-1}^{0} (g_0(n-k))(s)\, ds \\
&\quad + \sum_{i=0}^{k-1} (f_0(n-i))(i-k) = (f_0(n-k))(0) \\
&\quad + \sum_{i=0}^{k-1} (f_0(n-i))(i-k) = \sum_{i=0}^{k} (f_0(n-i))(i-k),
\end{aligned}
$$

$$
\begin{aligned}
(w(n))(-k-0) &= V(n-k)\, V^{-1}(n-k-1) \int_{0}^{0} (g_0(n-k-1))(s)\, ds \\
&\quad + \sum_{i=0}^{k} (f_0(n-i))(i-k) = \sum_{i=0}^{k} (f_0(n-i))(i-k).
\end{aligned}
$$

This computation also yields

$$
(w(n))(0) = (f_0(n))(0), \qquad n = 0, 1,\ldots . \tag{6.15}
$$

By Eqs. (6.13) and (6.14),

$$
w(0) = 0. \tag{6.16}
$$

Now $[t+1] = [t] + 1$, $-p \leqslant t \leqslant -1$, and therefore Eq. (6.14) yields, with Eq. (6.13),

$$
\begin{aligned}
&(w(n-1))(t+1) \\
&\quad = V(t+n)\, V^{-1}([t]+n) \int_{t-[t]-1}^{0} (g_0(n+[t]))(s)\, ds \\
&\qquad + \sum_{i=0}^{-[t]-2} (f_0(n-1-i))(t+1+i) \\
&\quad = (w(n))(t) - \sum_{i=0}^{-[t]-1} (f_0(n-i))(t+i) + \sum_{i=1}^{-[t]-1} (f_0(n-i))(t+i) \\
&\quad = (w(n) - f(n))(t), \qquad -p \leqslant t \leqslant -1, \quad n = 1, 2,\ldots .
\end{aligned} \tag{6.17}
$$

Finally, Eqs. (6.14) and (6.12) with Eqs. (6.13) and (5.1) yield

$$
\begin{aligned}
\|(w(n))(t)\| &\leqslant \| g_0(n+[t]) \| \exp |L| + \sum_{i=0}^{-[t]-1} \| f_0(n-i) \| \\
&\leqslant (\tfrac{3}{2} + |\hat{M}|)\, |f| \exp |L| + p\, |f|,
\end{aligned}
$$

so that w is bounded and Eq. (6.9) holds, with $k_2 = p + (\frac{3}{2} + |\hat{M}|) \exp|L|$.

In order to construct r we define $h \in \mathbf{s}_{[1]}(\mathbf{E}_1)$ by

$$(h(n))(t) = -(\hat{M}w(n-1))(t) + V(t+n)V^{-1}(n-1)(g_0(n-1))(t),$$
$$-1 \leqslant t \leqslant 0, \qquad n = 1, 2, \ldots . \quad (6.18)$$

Now Eqs. (6.10) and (6.13) imply, for $n = 1, 2, \ldots$,

$$\begin{aligned}(h(n))(0) &= -(\hat{M}w(n-1))(0) + V(n)V^{-1}(n-1)(g_0(n-1))(0)\\ &= -(\hat{M}(n-1)w(n-1))(0),\\ (h(n+1))(-1) &= -(\hat{M}w(n))(-1) + V(n)V^{-1}(n)(g(n))(-1)\\ &= -(\hat{M}(n)(w(n) - f(n)))(-1).\end{aligned} \quad (6.19)$$

By Eq. (6.17), $v_1 = w(n-1)$ and $v_2 = w(n) - f(n)$ satisfy $v_1(t+1) = v_2(t)$, $-p \leqslant t \leqslant -1$; by Lemma 4.1 (formula (4.2)) the rightmost sides of Eq. (6.19) are equal; therefore, so are the leftmost sides, and there exists, by Lemma 3.1, $r \in \mathbf{K}(E)$ with

$$\varpi r = h. \quad (6.20)$$

From Eqs. (6.18) and (6.20) we find, using Eq. (6.12) with Eqs. (6.13), (5.1), and (6.9),

$$\begin{aligned}\| \varpi r(n)\| &\leqslant \|\hat{M}(n-1)\| \, \| w(n-1)\| + \| g_0(n-1)\| \exp|L|\\ &\leqslant |\hat{M}| k_2 |f| + (\tfrac{3}{2} + |\hat{M}|)\, |f| \exp|L|.\end{aligned}$$

By Lemma 3.1, $r \in \mathbf{C}(E)$ and Eq. (6.7) follows, with

$$k_1 = p|\hat{M}| + (\tfrac{3}{2} + |\hat{M}|)(1 + |\hat{M}|) \exp|L|.$$

Now r and w, as constructed, satisfy Eqs. (6.7), (6.9), and (6.16). It therefore only remains to prove that w is a solution of Eq. (6.8). But Eqs. (6.1), (6.2), and (6.17) show that

$$\begin{aligned}(w(n) + A(n)w(n-1))(t) &= (w(n))(t) - (w(n-1))(t+1) = (f(n))(t)\\ &= (f(n) - B\varpi r(n))(t),\\ &\qquad -p \leqslant t \leqslant -1, \qquad n = 1, 2, \ldots ,\end{aligned}$$

so that Eq. (6.8) remains to be verified for $-1 \leqslant t \leqslant 0$ only. We use

Eqs. (6.14) (with continuity of $w(n)$ at $t = 0$), (6.1), (6.15), and (6.11) with Eq. (6.13) in turn to obtain, for $-1 \leqslant t \leqslant 0$, $n = 1, 2, \ldots$,

$$\begin{aligned}
&(w(n) + A(n)\, w(n-1))(t) \\
&\quad = V(t+n)\, V^{-1}(n-1) \int_t^0 (g_0(n-1))(s)\, ds + (f(n))(t) \\
&\qquad - V(t+n)\, V^{-1}(n-1)(w(n-1))(0) \\
&\qquad + \int_{-1}^t V(t+n)\, V^{-1}(s+n)(\hat{M} w(n-1))(s)\, ds \\
&\quad = (f(n))(t) + V(t+n)\, V^{-1}(n-1) \left(\int_t^0 (g_0(n-1))(s) \right) ds \\
&\qquad - (f_0(n-1))(0)) + \int_{-1}^t V(t+n)\, V^{-1}(s+n)(\hat{M} w(n-1))(s)\, ds \\
&\quad = (f(n))(t) - V(t+n)\, V^{-1}(n-1) \int_{-1}^t (g_0(n-1))(s)\, ds \\
&\qquad + \int_{-1}^t V(t+n)\, V^{-1}(s+n)(\hat{M} w(n-1))(s)\, ds.
\end{aligned} \tag{6.21}$$

But Eqs. (6.18), (6.20), and (6.2) yield, for the same t, n,

$$\begin{aligned}
&(f(n) - B\varpi r(n))(t) \\
&\quad = (f(n))(t) + \int_{-1}^t V(t+n)\, V^{-1}(s+n)(\hat{M} w(n-1))(s)\, ds \\
&\qquad - \int_{-1}^t V(t+n)\, V^{-1}(s+n)\, V(s+n)\, V^{-1}(n-1)(g_0(n-1))(s)\, ds;
\end{aligned} \tag{6.22}$$

and the rightmost sides of Eqs. (6.21) and (6.22) are plainly equal; so are then the leftmost sides, and the verification of Eq. (6.8) is complete.

7. Admissibility and the Solutions of the Homogeneous Equation

The purpose of the discussion of the preceding section is to enable us to replace consideration of the differential equations with delay, i.e., Eqs. (1.1) and (1.2), by analysis of the associated difference Eqs. (6.5) and (6.6), to which the theory in Ref. [1] can be applied. We shall have to rely on that paper for the crucial steps in the proofs. The assumptions of Sections 5 and 6 about L and M are in force, and A, B are defined by Eqs. (6.1) and (6.2).

We begin by considering the nonhomogeneous Eqs. (1.1) and (6.5). We say that $(\mathbf{C}, \mathbf{C})$ *is admissible with respect to* L, M—more loosely, *with respect to* Eq. (1.1)—if for every $r \in \mathbf{C}(E)$ there is a bounded solution u of Eq. (1.1). Similarly, $(\mathbf{l}^\infty, \mathbf{l}^\infty)$ *is admissible with respect to* A, or *with respect to* Eq. (6.5), if for every $f \in \mathbf{l}_{[1]}^{\infty}(\mathbf{E}_p)$ there exists a bounded solution x of Eq. (6.5) (see Ref. [1, p. 154]).

THEOREM 7.1. *$(\mathbf{C}, \mathbf{C})$ is admissible with respect to L, M if and only if $(\mathbf{l}^{\infty}, \mathbf{l}^{\infty})$ is admissible with respect to A.*

Proof. 1. Assume that $(\mathbf{C}, \mathbf{C})$ is admissible with repect to L, M. Let $f \in \mathbf{l}_{[1]}^{\infty}(\mathbf{E}_p)$ be given, and choose r and w as provided by Theorem 6.2. Since $r \in \mathbf{C}(E)$, there exists, by assumption, a bounded solution u of Eq. (1.1). Now Πu is bounded, and by Lemma 6.1 we have $\Pi u(n) + A(n)\Pi u(n-1) = B\varpi r(n)$, $n = 1, 2, \ldots$; since w is a bounded solution of Eq. (6.8), we conclude that $x = \Pi u + w$ is a bounded solution of Eq. (6.5). Thus $(\mathbf{l}^{\infty}, \mathbf{l}^{\infty})$ is admissible with respect to A.

2. Assume, conversely, that $(\mathbf{l}^{\infty}, \mathbf{l}^{\infty})$ is admissible with respect to A, and let $r \in \mathbf{C}(E)$ be given. By Eq. (6.4), $B\varpi r \in \mathbf{l}_{[1]}^{\infty}(\mathbf{E}_p)$; by the assumption there exists a bounded solution x of $x(n) + A(n)x(n-1) = B\varpi r(n)$, $n = 1, 2, \ldots$. By Lemma 6.1, $x = \Pi u$, where u is a solution of Eq. (1.1); and u is bounded. Thus $(\mathbf{C}, \mathbf{C})$ is admissible with respect to L, M.

In the theory of linear difference equations developed in Ref. [1], the following is a typical result: The admissibility of $(\mathbf{l}^{\infty}, \mathbf{l}^{\infty})$ with respect to the inhomeogeneous Eq. (6.5) implies, under certain additional conditions, a very special type of behavior of the solutions of the homogeneous Eqs. $(6.6)_{[m]}$, called an *exponential dichotomy* [1, Section 7]; roughly speaking, the bounded solutions tend uniformly exponentially to 0; there exists a "complementary" manifold of solutions of Eq. (6.6) tending uniformly exponentially to infinity; the two kinds of solutions stay apart; and together they span all solutions. Since Lemma 6.1 provides a bijective correspondence between solutions of Eqs. $(1.2)_{[m]}$ and $(6.6)_{[m]}$, Theorem 7.1 will allow us to translate that result into an analogous implication for differential equations with delays. We shall restrict ourselves, however, to finite-dimensional E; this restriction permits us, via a compactness argument (Lemma 7.2 and results in Ref. [6]), to gain a considerably sharpened insight without additional assumptions. Observe, though, that the difference equations (6.5) and (6.6) belong in the infinite-dimensional space $\mathbf{E}_p$.

With reference to Eqs. (6.5) and (6.6) we define, as usual, the transition operators $U(n, n_0) \in \mathbf{E}_p$ for integers $n \geqslant n_0 \geqslant 0$ as follows:

$$U(n_0, n_0) = I, \qquad U(n, n_0) = (-1)^{n-n_0} A(n)\, A(n-1) \cdots A(n_0+1), \qquad n > n_0 \geqslant 0. \tag{7.1}$$

LEMMA 7.2. *If E is finite-dimensional, $U(m+p, m)$ is a compact operator for $m = 0, 1, \ldots$.*

Proof. By Eqs. (6.1) and (6.2), $A(n) = -J + K(n) + B(n)\hat{M}(n-1)$, $n = 1, 2, \ldots$, where J, $K(n) \in \tilde{\mathbf{E}}_p$ are given by

$$(Jv)(t) = \begin{cases} v(t+1) - v(0), & -p \leqslant t \leqslant -1 \\ 0, & -1 \leqslant t \leqslant 0 \end{cases}$$

$$(K(n)\, v)(t) = \begin{cases} -v(0), & -p \leqslant t \leqslant -1 \\ -V(t+n)\, V^{-1}(n-1)\, v(0), & -1 \leqslant t \leqslant 0 \end{cases}$$

for all $v \in \mathbf{E}_p$. Now E is finite-dimensional; therefore $K(n)$ has finite rank, and hence it is compact. By Eq. (6.2), $B(n)$ is also compact in this case. Thus $A(n) + J$ is compact. It follows from Eq. (7.1) that $U(m+p, m) - J^p$ is compact. But induction shows that $(J^k v)(t) = 0$ for $-k \leqslant t \leqslant 0$, $k = 1, \ldots, p$, $v \in \mathbf{E}_p$; thus $J^p = 0$, and $U(m+p, m)$ is compact.

We now state our main result.

THEOREM 7.3. *Assume that E is finite-dimensional, and that $(\mathbf{C}, \mathbf{C})$ is admissible with respect to L, M. Then there exist numbers ν, $N > 0$ such that, for every $m \in \omega$, every bounded solution v of Eq. $(1.2)_{[m]}$ satisfies*

(i) $\| \Pi v(n) \| \leqslant N e^{-\nu(n-n_0)} \| \Pi v(n_0) \|$, $n \geqslant n_0 \geqslant m$;

there further exists a finite-dimensional linear manifold $\mathbf{W}$ of solutions of Eq. (1.2), and numbers ν', $N' > 0$, $\lambda_0 > 1$ such that, for every $m \in \omega$, every solution u of $(1.2)_{[m]}$ is of the form $u = v + w_{[m-p]}$, where v is a bounded solution and $w \in \mathbf{W}$, and such that every solution $w \in \mathbf{W}$ satisfies

(ii) $\| \Pi w(n) \| \geqslant N'^{-1} e^{\nu'(n-n_0)} \| \Pi w(n_0) \|$, $n \geqslant n_0 \geqslant 0$,

(iii) $\| \Pi w(n) \| \leqslant \lambda_0 \| \Pi w(n) - \Pi v(n) \|$, $n \geqslant m$,

v any bounded solution of Eq. $(1.2)_{[m]}$, $m \in \omega$.

Proof. By Theorem 7.1, $(\mathbf{l}^\infty, \mathbf{l}^\infty)$ is admissible with respect to A. To deal with Eqs. (6.5) and $(6.6)_{[m]}$, we now call upon Ref. [6], in the terminology and notation of Ref. [1]. Specifically, the condition (c) of Ref. [6, Lemma 4.2] is satisfied with $\mathbf{b} = \mathbf{d} = \mathbf{l}^\infty$; since $U(p, 0)$, say, is compact by Lemma 7.2, we conclude from Ref. [6, Theorem 4.3, (b)] that the covariant sequence $(\mathbf{E}_p)_0$ is regular and that its terms (the sets of initial values of the bounded solutions of Eq. $(6.6)_{[m]}$, $m = 0, 1, \ldots$) have constant finite co-dimension in $\mathbf{E}_p$. We can therefore apply the fundamental "direct" result [1, Theorem 10.2] to find that this covariant sequence induces an exponential dichotomy for A.

2. It remains to translate this knowledge about the solutions of Eq. $(6.6)_{[m]}$ by means of Lemma 6.1 into the conclusion of the theorem. (For a sketch of a proof of the converse implication, see Ref. [2, Theorem 8.4].)

We use the exponential dichotomy induced by $(\mathbf{E}_p)_0$ in the equivalent form given in condition (c) of Ref. [1, Theorem 7.1]; since $(\mathbf{E}_p)_0(0)$ has finite co-dimension in $\mathbf{E}_p$, the splitting in that statement may be taken to be a (linear) projection onto a finite-dimensional complementary subspace Z. We define $\mathbf{W}$ to be the finite-dimensional linear manifold of solutions w of Eq. (1.2) with $\Pi w(0) \in Z$. Then the conclusion of our theorem is simply a restatement of Ref. [1, Theorem 7.1 (c)] by means of Lemma 6.1.

Remark. Conversely, the conclusion of Theorem 7.3 implies that $(\mathbf{C}, \mathbf{C})$ is admissible with respect to L, M. This is only a special case of a much more general "converse" result [2; Theorems 8.6 and 8.4], which is itself a translation of a "converse" theorem for difference equations. Since this translation is quite straightforward and requires no fresh insight, we do not discuss it here.

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